# A Solution to Problem 19-1 in IMAGE No. 19

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## Abstract

A solution is offered for an eigenvalue problem posed in the Problem Corner of IMAGE, Number 19.

Key words

Eigenvalue.

IMAGE, the Bulletin of the International Linear Algebra Society, regularly publishes problems (and subsequently solutions) in matrix and linear algebra. Issue number 19, summer/fall 1997, contains the following problem, proposed by Jürgen Gro$\beta$ and Götz Trenkler of the Universität Dortmund, Germany.

**Problem 19-1.** Let $\mathbf{a}$ and $\mathbf{b}$ be two nonzero $n \times 1$ vectors and consider the matrix

$$\mathbf{A} = \alpha\mathbf{aa}' + \beta\mathbf{ab}' + \gamma\mathbf{ba}' + \delta\mathbf{bb}'$$

where $\alpha$, $\beta$, $\gamma$ and $\delta$ are real scalars such that $\beta = -\gamma$ and $\gamma^2 = -\alpha\delta$. Find the (nonzero) eigenvalues of $\mathbf{A}$.

## Solution

By conjecturing that $\mathbf{A} = \mathbf{w}_1\mathbf{w}_2'$ where $\mathbf{w}_i = h_i\mathbf{a} + k_i\mathbf{b}$ for $i = 1, 2$, where $h_i$ and $k_i$ are scalars, it is easily ascertained that

$$\begin{aligned}\mathbf{A} &= \left(\frac{-\gamma}{m_2}\mathbf{a} + \frac{\delta}{m_2}\mathbf{b}\right)\left(-\frac{m_2\alpha}{\gamma}\mathbf{a} + m_2\mathbf{b}\right)' \\ &= (-\gamma\mathbf{a} + \delta\mathbf{b})\left(-\frac{\alpha}{\gamma}\mathbf{a} + \mathbf{b}\right)'.\end{aligned}$$

Hence

$$\mathbf{A} = \mathbf{t}_1\mathbf{t}_2' \quad \text{for} \quad \mathbf{t}_1 = -\gamma\mathbf{a} + \delta\mathbf{b} \quad \text{and} \quad \mathbf{t}_2 = -\frac{\alpha}{\gamma}\mathbf{a} + \mathbf{b}.$$

Now suppose that $\lambda$ is a nonzero eigenvalue of $\mathbf{A}$, and $\mathbf{u}$ its associated eigenvector, so that $\mathbf{A}\mathbf{u} = \lambda\mathbf{u}$. Hence

$$\mathbf{t}_1\mathbf{t}_2'\mathbf{u} = \lambda\mathbf{u} \tag{1}$$

and so

$$\mathbf{t}_2'\mathbf{t}_1\mathbf{t}_2'\mathbf{u} = \lambda\mathbf{t}_2'\mathbf{u} \tag{2}$$

Since $\lambda$ is defined as being nonzero, and $\mathbf{u}$ is non-null (because being null would be meaningless), it follows from (1) that $\mathbf{t}_2'\mathbf{u}$ is nonzero. Therefore (2) reduces to $\mathbf{t}_2'\mathbf{t}_1 = \lambda$. Thus, with $\mathbf{A}$ having rank 1, its only nonzero eigenvalue is

$$\lambda = \mathbf{t}_2'\mathbf{t}_1 = \left(-\frac{\alpha}{\gamma}\mathbf{a} + \mathbf{b}\right)'(-\gamma\mathbf{a} + \delta\mathbf{b}) = \alpha\mathbf{a}'\mathbf{a} + \beta\mathbf{b}'\mathbf{a} + \gamma\mathbf{a}'\mathbf{b} + \delta\mathbf{b}'\mathbf{b}.$$